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HOMOGENIZATION OF THE PROCESS OF PHASE TRANSITIONS IN MULTIDIMENSIONAL HETEROGENEOUS PERIODIC MEDIA

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The homogenization of the Stefan multidimensional problem is carried out in the case where the medium is a composite consisting of two different substances with an ε-periodic structure. The averaged problem is deduced by asymptotic methods. It is shown that its solution is the limit of solutions of ε-problems.

The majority of composite materials have a periodic structure or a structure close to it and, therefore, in the present work the processes of phase transitions in multidimensional heterogeneous media with a periodic structure are studied. By the medium with a periodic structure, a medium composed of a periodically repeating element (cell) is understood. In this study, the homogenization of the Stefan multidimensional problem for a composite material is considered. Different approaches to the solution of this problem are proposed in [1].

Figure 1 shows one of the possible structures. The filled region A_ε is occupied by the substance A, and the light region B_ε by the substance B. The periodic cell is a cube $(0,\varepsilon)^k$ in $\mathbb{R}^k$ with the face ε (k is the dimensionality of spatial variables). The interface of the regions A_ε and B_ε is assumed to be a smooth function of the class C^l ($l > 2$). It is also assumed that the motion is absent and the densities of the substances A and B do not vary as the temperature changes. Under the action of a thermal field, the substances A and B of a composite material can undergo phase transformations with their melting point in accordance with the equation of state.

The phase transitions in each substance are described by the Stefan problem. We consider that in the regions A_ε and B_ε, the temperature $\theta_\varepsilon(x,t)$ satisfies, in terms of the distribution theory, the equations

$$\begin{aligned}
&\frac{\partial U_A(\theta_\varepsilon(x,t))}{\partial t} - \Delta\theta_\varepsilon(x,t) = f(x,t), \qquad x = (x_1,\ldots,x_k) \in A_\varepsilon,\\
&\frac{\partial U_B(\theta_\varepsilon(x,t))}{\partial t} - \Delta\theta_\varepsilon(x,t) = f(x,t), \qquad x = (x_1,\ldots,x_k) \in B_\varepsilon.
\end{aligned} \tag{1}$$

Here x is the spatial variables and t is the time; the subscript ε refers to the size of the periodic cell.

The qualitative dependence of the specific internal energies U_A and U_B on the temperature θ is shown in Fig. 2 (solid curves). The function $U_A(\theta)$ undergoes a discontinuity of the first kind for $\theta = \theta_A^*$, where θ_A^* is the melting point of the substance A; $U_A(\theta_A^* + 0) - U_A(\theta_A^* - 0) = L_A > 0$ is the latent heat of melting of the substance A. The function $U_B(\theta)$ undergoes a discontinuity of the first kind for $\theta = \theta_B^*$, where θ_B^* is the melting point of the substance B; $U_B(\theta_B^* + 0) - U_B(\theta_B^* - 0) = L_B > 0$ is the latent heat of melting of the substance B. Outside of the point of discontinuity, $U_A(\theta)$ and $U_B(\theta)$ are assumed to be increasing smooth functions:

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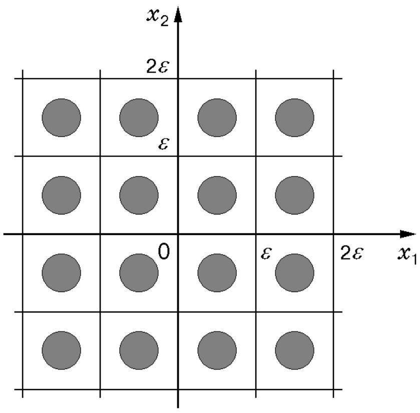

Fig. 1

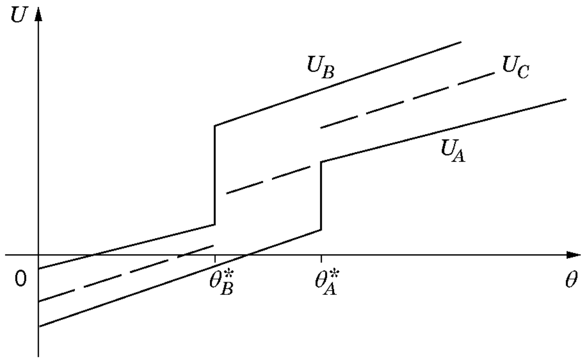

Fig. 2

$$U_A \in C^2[0, \theta_A^*], \quad U_A \in C^2[\theta_A^*, +\infty), \quad \frac{dU_A(\theta)}{d\theta} \geqslant c_0 \quad \text{for} \quad \theta \neq \theta_A^*,$$

$$U_B \in C^2[0, \theta_B^*], \quad U_B \in C^2[\theta_B^*, +\infty), \quad \frac{dU_B(\theta)}{d\theta} \geqslant c_0 \quad \text{for} \quad \theta \neq \theta_B^*.$$

Here $c_0 = \text{const} > 0$. The function inverse to $U_A(\theta)$ is denoted by θ_A [$\theta = \theta_A(U_A)$], and the function inverse to $U_B(\theta)$ is denoted by θ_B [$\theta = \theta_B(U_B)$]. We note that θ_A and θ_B are uniquely determined functions.

The following continuity conditions for the temperature and heat flux are satisfied on the surface of contact of the substances A and B:

$$[\theta_\varepsilon] = 0, \qquad \left[\frac{\partial \theta_\varepsilon}{\partial \boldsymbol{n}}\right] = 0. \tag{2}$$

Here $\boldsymbol{n}$ is the normal to the surface of contact; the square brackets denote the jump in the function in passing through the surface of contact.

We introduce the function

$$u(x, \theta) = \begin{cases} U_A(\theta), & x \in A_\varepsilon, \\ U_B(\theta), & x \in B_\varepsilon. \end{cases}$$

Using the periodic structure of the medium, we can consider that

$$u(x, \theta) = U(x/\varepsilon, \theta) = U(\xi, \theta),$$

where $\xi = x/\varepsilon$, $U(\xi, \theta)$ is a one-periodic function in ξ_i [$\xi_i = x_i/\varepsilon$ $(i = 1, \ldots, k)$]. Hereinafter, the term T-periodicity means the periodicity with period T in the indicated variables.

Let $\Omega \subset \mathbb{R}^k$ be the limited region with a smooth boundary S of the class C^l $(l > 2)$. Relations (1) and (2) in the region $\Omega_T = \Omega \times (0, T)$ are equivalent to the equation

$$\frac{\partial U(x/\varepsilon, \theta_\varepsilon(x, t))}{\partial t} - \Delta \theta_\varepsilon(x, t) = f(x, t), \qquad (x, t) \in \Omega_T \tag{3}$$

satisfied in terms of the distribution theory. At the boundary S of the region Ω, the temperature

$$\theta_\varepsilon(x, t) = \theta_S(x, t), \qquad (x, t) \in S_T = S \times (0, T) \tag{4}$$

is set, and the function $U_\varepsilon(x)$

$$U\Big|_{t=0} = U_\varepsilon(x) \qquad (x \in \Omega) \tag{5}$$

is defined at the initial moment of time. Using the given function $U_\varepsilon(x)$, one can find uniquely the initial temperature

$$\theta_\varepsilon\Big|_{t=0} = \begin{cases} \theta_A(U_\varepsilon(x)), & x \in A_\varepsilon \cap \Omega, \\ \theta_B(U_\varepsilon(x)), & x \in B_\varepsilon \cap \Omega. \end{cases} \tag{6}$$

Definition 1. The pair of functions $\{\theta_\varepsilon(x,t), u_\varepsilon(x,t)\}$ is called the generalized solution of the ε-problem (3)–(5) if:

1) $\theta_\varepsilon \in W_2^{1,0}(\Omega_T)$ and $u_\varepsilon(x,t) \in U(x/\varepsilon, \theta_\varepsilon(x,t))$;

2) the boundary condition (4) is satisfied;

3) for all the functions $\varphi(x,t)$ from $W_2^{1,1}(\Omega_T)$ subject to the conditions $\varphi\Big|_{t=T} = 0$ and $\varphi\Big|_{S_T} = 0$, the integral identity

$$\int\limits_{\Omega_T} u_\varepsilon \frac{\partial \varphi}{\partial t}\, dx\, dt - \int\limits_{\Omega_T} \nabla\theta_\varepsilon \nabla\varphi\, dx\, dt + \int\limits_{\Omega_T} f\varphi\, dx\, dt + \int\limits_{\Omega} U_\varepsilon(x)\varphi(x,0)\, dx = 0$$

is satisfied.

For convenience of the formulation of the result, we introduce the function $\theta_{\Gamma_\varepsilon}(x,t)$ which is defined in $\bar{\Omega} \times [0,T]$ and coincides with $\theta_S(x,t)$ from (4) for $(x,t) \in S_T$ and with $\theta_\varepsilon\Big|_{t=0}$ from (6) for $t=0$.

Theorem 1. *Let $\theta_{\Gamma_\varepsilon}(x,0) \in L_\infty(\Omega) \cap W_2^1(\Omega)$, $\theta_{\Gamma_\varepsilon} \in L_\infty(\Omega_T)$, $D_t\theta_{\Gamma_\varepsilon} \in W_2^{1,1}(\Omega_T)$, and $f \in L_2(\Omega_T)$. Then, there exists a unique limited generalized solution of problem* (3)–(5)

$$\theta_\varepsilon \in W_2^{1,1}(\Omega_T) \cap L_\infty(0,T; W_2^1(\Omega)) \cap L_\infty(\Omega_T), \qquad u_\varepsilon \in L_\infty(\Omega_T),$$

and the estimates on θ_ε and u_ε in the indicated classes being uniform in ε.

The proof of Theorem 1 is similar to that of the corresponding theorem for the generalized solution of the Stefan problem [2, pp. 32–35]. We note also that the existence of the generalized solution of a (3)–(5)-type problem is proved in [3].

We study the behavior of the solution $\theta_\varepsilon(x,t)$ as $\varepsilon \to 0$. Following to the general scheme [4], we search for an asymptotic solution of problem (3)–(5) in the form

$$\begin{gathered} \theta_\varepsilon(x,t) = \theta_0(x,t,\xi) + \varepsilon\theta_1(x,t,\xi) + \varepsilon^2\theta_2(x,t,\xi) + \ldots, \\ U(\xi, \theta_\varepsilon(x,t)) = U(\xi, \theta_0(x,t,\xi)) + \varepsilon\, \frac{\partial U(\xi, \theta_0(x,t,\xi))}{\partial \theta}\, \theta_1(x,t,\xi) + \ldots, \end{gathered} \tag{7}$$

where $\xi = x/\varepsilon$ and $\theta_j(x,t,\xi)$ are functions which are one-periodic in ξ. Substituting (7) into (3), we have

$$\begin{gathered} -\frac{1}{\varepsilon^2}\,(\Delta_\xi\theta_0(x,t,\xi)) - \frac{1}{\varepsilon}\biggl(\sum_{i=1}^{k}\Bigl(\frac{\partial^2\theta_0}{\partial x_i\,\partial\xi_i} + \frac{\partial^2\theta_0}{\partial\xi_i\,\partial x_i}\Bigr) + \Delta_\xi\theta_1(x,t,\xi)\biggr) + \frac{\partial U(\xi,\theta_0(x,t,\xi))}{\partial t} \\ - \Delta_x\theta_0(x,t,\xi) - \Delta_\xi\theta_2(x,t,\xi) - \sum_{i=1}^{k}\Bigl(\frac{\partial^2\theta_1}{\partial x_i\,\partial\xi_i} + \frac{\partial^2\theta_1}{\partial\xi_i\,\partial x_i}\Bigr) = f(x,t) + \varepsilon r(x,t,\xi). \end{gathered} \tag{8}$$

Equating terms of the order ε^{-2} to zero in (8), we obtain

$$\Delta_\xi\theta_0(x,t,\xi) = 0, \qquad \xi \in Q_1 = (0,1)^k, \tag{9}$$

where Q_1 is a unit cube in $\mathbb{R}^k$. We multiply (9) by $\theta_0(x,t,\xi)$ and integrate over ξ on the cube Q_1:

$$0 = \int\limits_{Q_1} \theta_0\Delta_\xi\theta_0\, d\xi_1\ldots d\xi_k = \int\limits_{Q_1} (\operatorname{div}_\xi(\theta_0\nabla_\xi\theta_0) - |\nabla_\xi\theta_0|^2)\, d\xi_1\ldots d\xi_k = \int\limits_{\partial Q_1} \theta_0(\nabla_\xi\theta_0\cdot\boldsymbol{n})\, ds - \int\limits_{Q_1} |\nabla_\xi\theta_0|^2\, d\xi_1\ldots d\xi_k.$$

The first term on the right side is equal to zero by virtue of the periodicity of the function θ_0 in the variables ξ. Therefore, $|\nabla_\xi\theta_0| = 0$. Hence,

$$\theta_0 = \theta_0(x,t), \tag{10}$$

i.e., θ_0 does not depend on the variable ξ.

Equating terms of the order ε^{-1} to zero in (8), with allowance for (10) we obtain $\Delta_\xi \theta_1(x,t,\xi) = 0$ ($\xi \in Q_1$), where $\theta_1(x,t,\xi)$ is a function which is one-periodic in ξ. As for Eq. (9), one can show that

$$\theta_1 = \theta_1(x,t), \tag{11}$$

i.e., the function θ_1 does not depend on the variable ξ.

Equating terms of the order ε^0 to zero in (8), with allowance for (10) and (11) we obtain

$$\frac{\partial U(\xi, \theta_0(x,t))}{\partial t} - \Delta_x \theta_0(x,t) = \Delta_\xi \theta_2(x,t,\xi) + f(x,t). \tag{12}$$

We introduce the average over the period:

$$\langle g(x_1,\dots,x_k,t,\xi_1,\dots,\xi_k)\rangle = \int_0^1 \dots \int_0^1 g(x_1,\dots,x_k,t,\xi_1,\dots,\xi_k)\, d\xi_1 \dots d\xi_k.$$

We apply the averaging operator to both sides of (12)

$$\Big\langle \frac{\partial U(\xi, \theta_0(x,t))}{\partial t} \Big\rangle - \langle \Delta_x \theta_0(x,t)\rangle = \langle \Delta_\xi \theta_2(x,t,\xi)\rangle + f(x,t). \tag{13}$$

Since the function θ_2 is periodic in ξ, we have

$$\langle \Delta_\xi \theta_2(x,t,\xi)\rangle = 0, \quad \langle \Delta_x \theta_0(x,t)\rangle = \Delta_x \theta_0(x,t), \quad \Big\langle \frac{\partial U}{\partial t}\Big\rangle = \frac{\partial}{\partial t}\langle U(\xi, \theta_0(x,t))\rangle.$$

As a result, we obtain the equation for determination of the function $\theta_0(x,t)$, which it is natural to call the averaged equation:

$$\frac{\partial U_C(\theta_0(x,t))}{\partial t} - \Delta_x \theta_0(x,t) = f(x,t). \tag{14}$$

Here $U_C(\theta_0(x,t)) \equiv \langle U(\xi, \theta_0(x,t))\rangle = v_A U_A(\theta_0(x,t)) + v_B U_B(\theta_0(x,t))$, where v_A and $v_B = 1 - v_A$ are the volumes occupied by the substances A and B, respectively, in the unit cube Q_1.

The qualitative dependence of the averaged specific internal energy $U_C(\theta)$ is shown in Fig. 2 (dashed curve). The strictly increasing function $U_C(\theta)$ undergoes discontinuities of the first kind for $\theta = \theta_B^*$ and $\theta = \theta_A^*$; here $U_C(\theta_B^* + 0) - U_C(\theta_B^* - 0) = v_B L_B > 0$ and $U_C(\theta_A^* + 0) - U_C(\theta_A^* - 0) = v_A L_A > 0$. Outside of the points of discontinuity, $U_C(\theta)$ is a smooth function. We denote the function inverse to $U_C(\theta)$ by θ_C, i.e., $\theta = \theta_C(U_C)$. The function θ_C is a uniquely determined function.

Let the function $\theta_0(x,t)$ satisfy, in terms of the distribution theory, Eq. (14) and the initial boundary conditions

$$\theta_0(x,t) = \theta_S(x,t), \qquad (x,t) \in S \times (0,T); \tag{15}$$

$$U_C\Big|_{t=0} = U_0(x), \qquad x \in \Omega, \tag{16}$$

where $U_0(x)$ is the $*$-weak limit of the functions $U_\varepsilon(x)$ in $L_\infty(\Omega)$ as $\varepsilon \to 0$. Using the function $U_0(x)$, one can find unambiguously the initial temperature

$$\theta_0\Big|_{t=0} = \theta_C(U_0(x)), \qquad x \in \Omega. \tag{17}$$

Definition 2. The pair of functions $\{\theta_0(x,t), u_0(x,t)\}$ is called the generalized solution of problem (14)–(16) if:

1) $\theta_0 \in W_2^{1,0}(\Omega_T)$ and $u_0(x,t) \in U_C(\theta_0(x,t))$;

2) the boundary condition (15) is satisfied;

3) for all the functions $\varphi(x,t)$ from $W_2^{1,1}(\Omega_T)$ subject to the conditions $\varphi\Big|_{t=T} = 0$ and $\varphi\Big|_{S_T} = 0$, the integral identity

$$\int\limits_{\Omega_T} \Big(u_0 \frac{\partial \varphi}{\partial t} - \nabla\theta_0 \nabla\varphi + f\varphi\Big) dx\, dt + \int\limits_{\Omega} U_0(x)\varphi(x,0)\, dx = 0$$

is satisfied.

For convenience of the formulation of the result, we introduce the function $\theta_{\Gamma_0}(x,t)$ which is defined in $\bar\Omega \times [0,T]$ and coincides with $\theta_S(x,t)$ from (15) for $(x,t) \in S_T$ and with $\theta_0\Big|_{t=0}$ from (17) for $t=0$.

Theorem 2. *Let $\theta_{\Gamma_0}(x,0) \in L_\infty(\Omega) \cap W_2^1(\Omega)$, $\theta_{\Gamma_0} \in L_\infty(\Omega_T)$, $D_t\theta_{\Gamma_0} \in W_2^{1,1}(\Omega_T)$, and $f \in L_2(\Omega_T)$. Then, there exists a unique limited generalized solution of problem* (14)–(16):

$$\theta_0 \in W_2^{1,1}(\Omega_T) \cap L_\infty(0,T; W_2^1(\Omega)) \cap L_\infty(\Omega_T), \qquad u_0 \in L_\infty(\Omega_T).$$

The proof of Theorem 2 is similar to that of the corresponding theorem for the generalized solution of the Stefan problem [2, pp. 32–35].

Theorem 3. *Let $\{\theta_\varepsilon, u_\varepsilon\}$ be the solution of the ε-problem* (3)–(5). *Then, there is a sequence* (*denoted by the subscript ε*) *$\{\theta_\varepsilon, u_\varepsilon\}$ which converges to the solution $\{\theta_0, u_0\}$ of the average problem* (14)–(16) *as $\varepsilon \to 0$ in the following meaning*: *$\theta_\varepsilon \to \theta_0$ is weak in $W_2^{1,1}(\Omega_T)$ and strong in $L_2(\Omega_T)$ and almost everywhere in Ω_T; $u_\varepsilon \to u_0$ is $*$-weak in $L_\infty(\Omega_T)$.*

Proving Theorem 3, we use the following lemma (its formulation and proof can be found, for example, in [5]).

Lemma 1. *Let D be a hypercube in $\mathbb{R}^k$ and the function $g(x) \in L_p(D)$ ($p \geqslant 1$) be continued periodically* (*in each variable*) *from D into $\mathbb{R}^k$. Then, the functions $g(x/\varepsilon)$ converge weakly to*

$$\langle g\rangle \equiv \frac{1}{\operatorname{meas} D} \int\limits_D g(x)\, dx \quad in \quad L_p(D) \quad as \quad \varepsilon \to 0.$$

If $p = +\infty$, then $g(x/\varepsilon)$ converge $*$-weakly to $\langle g\rangle$ in $L_\infty(D)$ as $\varepsilon \to 0$.

Proof of Theorem 3. It follows from the results of Theorem 1 that there is a sequence $\{\theta_\varepsilon\}$ which converges weakly to a certain function θ_0 in $W_2^{1,1}(\Omega_T)$, strongly in $L_2(\Omega_T)$, and almost everywhere in Ω_T as $\varepsilon \to 0$.

We consider the first integral on the left side of integral identity 3 from Definition 1:

$$\int\limits_0^T \int\limits_\Omega u_\varepsilon \varphi_t\, dx\, dt = \int\limits_0^T \int\limits_{\Omega\cap A_\varepsilon} U_A(\theta_\varepsilon(x,t))\varphi_t\, dx\, dt + \int\limits_0^T \int\limits_{\Omega\cap B_\varepsilon} U_B(\theta_\varepsilon(x,t))\varphi_t\, dx\, dt$$

$$= \int\limits_0^T \int\limits_\Omega \chi_{A_\varepsilon}\{U_A(\theta_\varepsilon(x,t)) - U_A(\theta_0(x,t))\}\varphi_t\, dx\, dt + \int\limits_0^T \int\limits_\Omega \chi_{A_\varepsilon} U_A(\theta_0(x,t))\varphi_t\, dx\, dt$$

$$+ \int\limits_0^T \int\limits_\Omega \chi_{B_\varepsilon}\{U_B(\theta_\varepsilon(x,t)) - U_B(\theta_0(x,t))\}\varphi_t\, dx\, dt + \int\limits_0^T \int\limits_\Omega \chi_{B_\varepsilon} U_B(\theta_0(x,t))\varphi_t\, dx\, dt. \tag{18}$$

Here $\chi_{A_\varepsilon}(x)$ and $\chi_{B_\varepsilon}(x)$ are characteristic functions of the sets A_ε and B_ε; we note that $\chi_{A_\varepsilon}(x) = \chi_{A_1}(x/\varepsilon)$ and $\chi_{B_\varepsilon}(x) = \chi_{B_1}(x/\varepsilon)$. Since $\theta_\varepsilon(x,t) \to \theta_0(x,t)$ almost everywhere in Ω_T, the first and third integrals on the right side of (18) converge to zero (according to the Lebesgue theorem on the limit transition under the integral). The functions $U_A(\theta_0(x,t))\varphi_t(x,t)$ and $U_B(\theta_0(x,t))\varphi_t(x,t)$ in the second and fourth integrals do not depend on ε, and they can be considered as trial functions from $L_1(\Omega_T)$ ($\varphi_t \in L_1(\Omega_T)$, and $U_A(\theta_0)$ and $U_B(\theta_0)$ are limited in $L_\infty(\Omega_T)$ by virtue of the boundedness of θ_0 in $L_\infty(\Omega_T)$). Then, according to Lemma 1, χ_{A_ε} converge $*$-weakly to v_A in $L_\infty(\Omega)$ as $\varepsilon \to 0$ and χ_{B_ε} converge $*$-weakly to $v_B = 1 - v_A$ in $L_\infty(\Omega)$; this implies that as $\varepsilon \to 0$, the second and fourth integrals on the right side of (18) converge to $v_A \int\limits_{\Omega_T} U_A(\theta_0(x,t))\varphi_t(x,t)\,dx\,dt$ and $v_B \int\limits_{\Omega_T} U_B(\theta_0(x,t))\varphi_t(x,t)\,dx\,dt$. Thus,

$$\int\limits_{\Omega_T} u_\varepsilon \varphi_t\,dx\,dt \to \int\limits_{\Omega_T} u_0\varphi_t\,dx\,dt \quad \text{as} \quad \varepsilon \to 0,$$

where $u_0(x,t) = v_A U_A(\theta_0(x,t)) + v_B U_B(\theta_0(x,t))$.

The limit transition as $\varepsilon \to 0$ in integral identity 3 of Definition 1 leads to integral identity 3 of Definition 2. By virtue of the uniqueness of the solution of the average problem, the theorem is proved.

It follows from the theorem that the characteristics of the averaged problem depend on the ratio of the volumes v_A and v_B of the substances A and B but do not depend on their mutual geometrical position.

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